

The thermodynamics of self-gravitating systems in equilibrium is holographic

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Abstract

We prove that in any static spacetime the maximization of matter entropy subject to the continuity equation for the stress-energy tensor and the initial-value constraint leads to the full set of Einstein equations and to Tolman's relation between local temperature and lapse function. This result is shown to imply that the thermodynamics of gravitating systems in equilibrium is fully specified by variables defined on the system's boundary, in particular, the boundary's geometry and extrinsic curvature. Hence, self-gravitating systems in equilibrium are holographic already at the level of classical relativity.

The association of entropy to black holes by Bekenstein and Hawking strongly suggests of a fundamental relationship between gravity and thermodynamics. It has led to the conjecture that the fundamental theory of gravity should satisfy the holographic principle [1], namely, the statement that the full information about a gravitational system is contained in the degrees of freedom of the system's boundary.

In this letter, we show that self-gravitating systems in general relativity are holographic already at the level of the classical theory with no reference to black holes or the Bekenstein-Hawking entropy. We demonstrate that the thermodynamical properties of a self-gravitating system in equilibrium are defined solely in terms of variables defined on the system's boundary, in particular, the boundary's metric and extrinsic curvature.

To this end, we first prove that in a general static spacetime the full set of Einstein's equations follow from the maximization of matter entropy subject to (i) the continuity equation for the stress-energy tensor, (ii) the initial value (Hamiltonian) constraint and (iii) suitable boundary conditions. This proof generalizes previous results that were derived for spherically symmetric static spacetimes—see, Refs. [2, 3] for specific model systems, Ref. [4] for the most general spherically symmetric case, and Ref. [5] for elaborations.

The proof proceeds as follows. Consider a static globally hyperbolic spacetime $M = R \times \Sigma$ with four-metric

$$ds^2 = -L^2(x)dt^2 + h_{ij}(x)dx^i dx^j, \quad (1)$$

expressed in terms of the spatial coordinates x^i and the time coordinate t . The time-like vector field $\xi^\mu = (\partial/\partial t)^\mu$ is a Killing vector of the metric Eq. (1). L is the lapse function, and h_{ij} is a t -independent Riemannian three-metric on the surfaces Σ_t of constant t . The time-like unit normal on Σ_t is $n_\mu = L\partial_\mu t$ and the extrinsic curvature tensor on Σ_t vanishes.

Thermodynamic definitions. Let $V \subset \Sigma$ be a compact spatial region, with boundary $B = \partial V$. V contains an isotropic fluid in thermal and dynamical equilibrium, described by the stress-energy tensor

$$T_{\mu\nu} = \rho(x)n_\mu n_\nu + P(x)(g_{\mu\nu} + n_\mu n_\nu), \quad (2)$$

where $\rho(x)$ and $P(x)$ are the energy density and the pressure, respectively.

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The stress-energy tensor satisfies the continuity equation $\nabla_\mu T^{\mu\nu} = 0$ which implies that

$$\frac{\nabla_i P}{\rho + P} = -\frac{\nabla_i L}{L}. \quad (3)$$

Eq. (2) is the standard form of the stress-energy tensor for ideal fluids. Since dissipative processes are absent in equilibrium configurations, Eq. (2) also applies to non-ideal fluids.

We assume that the fluid consists of r different particle species. The associated particle-number densities $n_a(x)$, $a = 1, \dots, r$, together with the energy density $\rho(x)$ define the thermodynamic state space. All local thermodynamic properties of the fluid are encoded in the entropy-density functional $s(\rho, n_a)$. The first law of thermodynamics takes the form

$$Tds = d\rho - \sum_a \mu_a dn_a, \quad (4)$$

where $\mu_a = -T \frac{\partial s}{\partial n_a}$ is the chemical potential associated to particle species a and $T = \left(\frac{\partial s}{\partial \rho}\right)^{-1}$ is the local temperature. The pressure P is defined through the Euler equation

$$\rho + P - Ts - \sum_a \mu_a n_a = 0. \quad (5)$$

Combining Eqs. (5) and (4), we derive the Gibbs-Duhem relation, $dP = sdT + \sum_a n_a d\mu_a$.

Entropy maximization. Next, we maximize the total entropy $S = \int_V d^3x \sqrt{h} s(\rho, n_a)$ for fixed values of the particle numbers $N_a = \int_V d^3x \sqrt{h} n_a(\rho, n_a)$. Entropy maximization is subject to the continuity equation (3) for the fluid, and to the Hamiltonian constraint of general relativity. For static spacetimes, the Hamiltonian constraint reads

$$C(x) := 16\pi\rho(x) - R(x) = 0, \quad (6)$$

where R is the Ricci scalar associated to the three-metric h_{ij} .

The Hamiltonian constraint expresses the energy density ρ as a function of the metric h_{ij} and the continuity equation expresses the lapse L in terms of the energy density ρ . It follows that the entropy S and the particle numbers N_a are functionals of the three-metric h_{ij} and the particle-number densities n_a . Entropy maximization requires the variation of $S + \sum_a b_a N_a$, with respect to n_a and h_{ij} for some Lagrange multipliers b_a .

Variation of $S + \sum_a b_a N_a$ with respect to n_a yields,

$$\delta S = \sum_a \int d^3x \sqrt{h} \left(\frac{\partial s}{\partial n_a} + b_a \right) \delta n_a = \sum_a \int d^3x \sqrt{h} \left(-\frac{\mu_a}{T} + b_a \right) \delta n_a = 0, \quad (7)$$

leading to $b_a = \frac{\mu_a}{T}$. Hence, for equilibrium configurations the thermodynamic variables $\frac{\mu_a}{T}$ are constant in V .

We introduce the function ω (a Massieu function [7]) as the Legendre transform of the entropy density s with respect to n_a

$$\omega(\rho, b_a) := s - \sum_a \frac{\partial s}{\partial n_a} n_a = s + \sum_a b_a n_a = \frac{\rho + P}{T}. \quad (8)$$

Expressed in terms of ω , Eq. (4) takes the form

$$d\omega = \frac{d\rho}{T} - \sum_a n_a db_a. \quad (9)$$

It follows that $T^{-1} = \partial\omega/\partial\rho$ and $n_a = -\partial\omega/\partial b_a$. The Gibbs-Duhem relation becomes $dP = \omega dT + T \sum_a n_a db_a$.

Tolman's relation. Since the variables b_a are constant for entropy-maximizing configurations, $dP/dT = \omega = (P + \rho)/T$ in V . Combining with Eq. (3), we obtain

$$\frac{\nabla_i T}{T} = -\frac{\nabla_i L}{L}, \quad (10)$$

which leads to Tolman's relation between local temperature and lapse function [6]

$$LT = T_*, \quad (11)$$

where T_* is a constant. In an asymptotically flat spacetime, $L = 1$ at spacelike infinity. Then, T_* is identified with the temperature as seen by an observer at infinity. Eq. (11) is the generalization of the zero-th law of thermodynamics: the temperature of a self-gravitating body in equilibrium is everywhere constant when transformed to the frame of an observer located at infinity. We note that the only assumptions required for the derivation of Tolman's relation are (i) that the entropy density s does not depend on the metric h_{ij} and (ii) that the Hamiltonian constraint does not involve the particle densities n_a . Tolman's relation is essentially kinematical, in the sense that its derivation does not depend on the explicit form of Einstein's equations. It follows from Galileo's principle that gravity is insensitive to the physical composition of material bodies.

Derivation of Einstein's equations. By employing the definition Eq. (8), entropy-maximization is equivalent to the maximization of the functional Ω , defined as

$$\Omega[h_{ij}, b_a] := \int d^3x \sqrt{h} \omega(\rho, b_a), \quad (12)$$

The variation of Ω with respect to metrics h_{ij} that satisfy the Hamiltonian constraint Eq. (6) yields

$$\delta\Omega = \int d^3x \sqrt{h} \left(\frac{\omega}{2} h^{ij} \delta h_{ij} + \frac{\partial \omega}{\partial \rho} \delta \rho \right) = \int d^3x \sqrt{h} \left(\frac{\rho + P}{2T} h^{ij} \delta h_{ij} + \frac{\delta R}{16\pi T} \right). \quad (13)$$

Using the equation $\delta R = -R^{ij} \delta h_{ij} + \nabla^i (\nabla^j \delta h_{ij} - h^{kl} \nabla_i \delta h_{kl})$ for the variation of the Ricci scalar R , together with Eq. (10), we find

$$\begin{aligned} \delta\Omega = - \int d^3x \frac{\sqrt{h}}{16\pi T} S^{ij} \delta h_{ij} + \frac{1}{16\pi} \int d^3x \sqrt{h} \left[\nabla^i (T^{-1} \nabla^j \delta h_{ij} - T^{-1} h^{kl} \nabla_i \delta h_{kl}) \right. \\ \left. - \nabla^j \left(\frac{\nabla^i L}{LT} \delta h_{ij} - \frac{\nabla_j L}{LT} h^{kl} \delta h_{kl} \right) \right], \end{aligned} \quad (14)$$

where

$$S^{ij} := R^{ij} - \frac{1}{2} h^{ij} R - \frac{1}{L} (\nabla^i \nabla^j L - h^{ij} \nabla_k \nabla^k L) - 8\pi h^{ij} P. \quad (15)$$

In order to compute the total divergence term in Eq. (14), we perform a $2 + 1$ decomposition of the metric h_{ij} on the bounding surface B . Let B be described locally by the condition $f(x) = 0$. The unit normal to B is $m_i = \alpha \nabla_i f$, where $\alpha = 1/\sqrt{h^{ij} \nabla_i f \nabla_j f}$. The induced metric on B is $\sigma_{ij} = h_{ij} - m_i m_j$ and the extrinsic curvature tensor of B is $\kappa_{ij} := \sigma_i^k \sigma_j^l \nabla_k m_l$.

Then, Eq. (14) becomes

$$\delta\Omega = - \int d^3x \sqrt{h} S^{ij} \delta h_{ij} + \frac{1}{16\pi T_*} \oint_B d^2y \sqrt{\sigma} \left(m^i \nabla_i L \sigma^{kl} \delta \sigma_{kl} - 2L \sigma^{ij} \delta \kappa_{ij} + L \kappa^{ij} \delta \sigma_{ij} \right), \quad (16)$$

where the coordinates on B are denoted as y .

For a variation with fixed values of the two-metric σ_{ij} and of the extrinsic curvature κ_{ij} on the boundary, $\delta\Omega = - \int d^3x \sqrt{h} S^{ij} \delta h_{ij}$. Thus, the maximum entropy principle, $\delta\Omega = 0$, leads to the condition $S^{ij} = 0$, which coincides the spatial components of Einstein's equations for the static spacetime metric, Eq. (1).

The equation $S^{ij} = 0$ corresponds to Hamilton's equations for the momentum π^{ij} , conjugate to h_{ij} [8]. In the Hamiltonian formulation of general relativity, the equations of motion incorporate the symplectic structure (Poisson bracket) and the constraints of the gravitational state space. The Hamiltonian constraint was employed in the derivation of Eq. (16). Hence, we conclude that the thermodynamic description of the system contains implicitly information about the symplectic structure of general relativity [9].

The thermodynamic state space of self-gravitating systems. Next, we employ Eq. (16) in order to formulate the thermodynamic properties of self-gravitating systems. To this end, we concentrate on solutions to Einstein's equations, $S^{ij} = 0$, and examine the changes to Ω under external variations of the boundary conditions.

Eq. (16) applies to any spatial region V and boundary B . When studying an isolated self-gravitating system, B is to be identified with the system's physical boundary, for example, a bounding box (as in ordinary thermodynamics) or a stellar surface. B separates between a region in which the metric h_{ij} is given by a solution of Einstein's equation with matter (the interior) and a region where h_{ij} is a solution of the vacuum Einstein equations (the exterior). Assuming that the metric h_{ij} is regular in the interior region (i.e., everywhere locally Minkowskian, no inner boundaries), then Eq. (16) leads to

$$\delta\Omega = \frac{1}{16\pi T_*} \oint_B d^2y \sqrt{\sigma} \left(m^i \nabla_i L \sigma^{kl} \delta\sigma_{kl} - 2L\sigma^{ij} \delta\kappa_{ij} + L\kappa^{ij} \delta\sigma_{ij} \right). \quad (17)$$

We observe that Ω varies only with the two-metric σ_{ij} and the extrinsic curvature κ_{ij} of the boundary. No other components of the three metric and its derivatives have thermodynamic significance. Ω also depends on the variables b_a ; these are constant in V , and thus their value is also determined at the boundary. It follows that the thermodynamic state space for a self-gravitating system in equilibrium consists only of variables that are accessible at the system's boundary, and that the function $\Omega[\sigma_{ij}, \kappa_{ij}, b_a]$, defined solely on boundary data, fully specifies the thermodynamic properties of the system.

Moreover, by Eq. (17) the thermodynamically conjugates $\delta\Omega/\delta\sigma_{ij}(s)$ and $\delta\Omega/\delta\kappa_{ij}(s)$ are also boundary variables. The particle numbers $N_a = -\frac{\partial\Omega}{\partial b_a}$ are bulk variables, however, they are conserved in a closed system, and their value can be ascertained from information about the process through which the system was formed. An external observer with no knowledge about the system, other than the conserved numbers N_a , can fully reconstruct the equations of state (the functional relation between state-space variables and their conjugates) solely by local measurements at the boundary. Thus, the thermodynamics of self-gravitating systems exhibit a striking manifestation of the holographic principle; all hydrodynamic bulk variables (density, pressure) are encoded into geometric properties of the boundary.

We must emphasize that the holographic description, derived here, refers only to thermodynamic properties of a self-gravitating system; not all bulk variables can be identified by boundary variables. This is the case in particular for the three metric h_{ij} on V . The thermodynamic variables σ_{ij} and κ_{ij} correspond to three components and three first derivatives of h_{ij} . Viewed as boundary data, they are not sufficient for finding a unique solution to Einstein's equations in the interior. Hence, many different solutions to Einstein's equations correspond to the same values of the thermodynamic variables on the boundary.

This point has been made in Ref. [10] for a particular class of spherically symmetric solutions corresponding to self-gravitating radiation in a box. It was shown that, nonetheless, the maximum entropy principle leads to a unique solution to Einstein's equations solely from the knowledge of the boundary variables. In this system, most solutions of Einstein's equation with fixed boundary condition contain a conical singularity at the origin; regular solutions form a set of measure zero in the space of all solutions. Including a contribution from the singularity to the total entropy of the systems allows for the implementation of the maximum entropy principle. Regular solutions turn out to maximize entropy for any fixed values of the thermodynamical variables on the boundary. We believe that the methodology of Ref. [10] can be generalized to a larger class of spacetimes, possibly to any self-gravitating system in equilibrium. This requires the extension of the maximum entropy principle also to non-regular spacetimes, taking into account the contribution of internal boundaries and singularities. This issue is presently under investigation.

Weak gravity limit. It is instructive to see how the correspondence between bulk-hydrodynamics and boundary geometry works at the limit of weak gravitational interaction. We assume that the boundary corresponds to a bounding box. For low densities and small system size, we expect to recover ordinary thermodynamics expressed in terms of extensive variables. Hence, the matter density ρ is approximately constant on the interior of the box. By Eq. (6), the Ricci scalar R will also be constant. Hence, the geometry inside the box will correspond to a region of a homogeneous three-sphere of radius $\sqrt{6/R}$. For simplicity, we assume a spherical boundary, so it is convenient to use coordinates (r, θ, ϕ) , in which the three-sphere metric reads

$$ds^2 = \frac{dr^2}{1 - \frac{Rr^2}{6}} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (18)$$

At the weak gravity limit, R is close to zero, so Eq. (18) approaches the Minkowskian metric. For a boundary B defined as $r = r_0$, σ_{ij} is the standard two-metric for a sphere of radius r_0 . The extrinsic curvature is $\kappa_{ij} \simeq \kappa_{ij}^{(0)} \left(1 - \frac{4\pi\rho r_0^2}{3}\right)$, where $\kappa_{ij}^{(0)}$ is the extrinsic curvature of the surface $r = r_0$ when embedded in Minkowski spacetime. Thus, the energy density of ordinary, extensive thermodynamics is proportional to the deviation of the extrinsic curvature from its Minkowskian value: $\rho = -\frac{3}{4\pi r_0^2}(\kappa - \kappa^{(0)})/\kappa^{(0)}$, where $\kappa = \sigma^{ij}\kappa_{ij}$. The function $\Omega \simeq \omega(\rho, b_a)(\frac{4\pi}{3}r_0^3)$ is expressed solely in terms of the geometric properties r_0 and $(\kappa - \kappa^{(0)})/\kappa^{(0)}$ of the boundary.

First law of thermodynamics. The projection of the equations of motion along m^i , $S_{ij}m^im^j = 0$, implies that

$$m^i\nabla_i L = \frac{L}{\kappa} \left[8\pi P + \frac{1}{L}\sigma_{ij}{}^2\nabla_i{}^2\nabla_j L + \frac{1}{2}(\kappa_{ij}\kappa^{ij} - \kappa^2 + {}^2R) \right]. \quad (19)$$

where 2R is the curvature scalar on B and ${}^2\nabla_i$ the covariant derivative associated to σ_{ij} . A natural thermodynamic assumption is that the local temperature is constant on B , so that there is no boundary heat flow. This implies that ${}^2\nabla_i L = 0$. Substituting Eq. (19) into Eq. (17), we obtain

$$\begin{aligned} \delta\Omega &= \frac{1}{T_*} \oint d^2s \frac{LP}{\kappa} \delta(\sqrt{\sigma}) - \sum_a N_a \delta b_a \\ &+ \frac{1}{16\pi T_*} \oint d^2y L \sqrt{\sigma} \left(\kappa^{ij} \delta\sigma_{ij} - 2\sigma^{ij} \delta\kappa_{ij} + \frac{\kappa_{ij}\kappa^{ij} - \kappa^2 + {}^2R}{2\kappa} \sigma^{kl} \delta\sigma_{kl} \right), \end{aligned} \quad (20)$$

where we included variations with respect to b_a . Eq. (20) is the first law of thermodynamics for a self-gravitating system. The surface integral involving P corresponds to mechanical work generated by variations of the boundary area. The heat term $T_*\delta S$ and the chemical potential terms $\sum_a \mu_a \delta N_a$ are readily obtained from $T_*(\delta\Omega + \sum_a N_a \delta b_a)$. However, the identification of internal energy is not obvious. Let us denote the term the second line of Eq. (20) by Θ/T_* ; Θ is an one-form on the thermodynamic state space. The exterior derivative of Θ does not vanish, so Θ is not a exact form. Hence, it cannot be identified with a variation of the internal energy; it also involves work terms.

Eq. (20) simplifies significantly for spherically symmetric systems. The exterior metric is Schwarzschild, by virtue of Birkhoff's theorem, and it is characterized by a value M of the Arnowitt-Deser-Misner (ADM) mass; σ_{ij} is the two-metric for a sphere of radius r_0 ; $\kappa_{ij} = \sqrt{1 - 2M/r_0}/r_0 \sigma_{ij}$, and $L = \sqrt{1 - 2M/r_0}$. Then, $\Theta = 16\pi\delta M$, and the first law of thermodynamics takes the transparent form

$$\delta M = T_*\delta S - P(r_0)(4\pi r_0^2 \delta r_0) + L(r_0) \sum_a \mu_a(r_0) \delta N_a. \quad (21)$$

In spherically symmetric systems, internal energy is identified with the ADM mass, and the one-form Θ contains no work terms. This suggests that, in the general case, the work terms in Θ correspond to deformation of boundary inhomogeneities.

The Komar mass. For static solutions to Einstein's equations, a natural candidate for the internal energy is the Komar mass M , defined in terms of the surface integral

$$M = \frac{1}{4\pi} \oint_{B'} d^2y \sqrt{\sigma} m^k \nabla_k L. \quad (22)$$

for any closed surface B' enclosing B , since the energy density and pressure vanishes in the exterior of B .

We will show that the Komar mass is closely related to the thermodynamic function Ω . For a gravitating system bound by a box, the pressure is discontinuous at B . The metric components and the derivatives corresponding to the thermodynamical variables σ_{ij} and κ_{ij} are continuous across the boundary, but the derivative $m^k \nabla_k L$ exhibits a jump, which we will denote as $[m^k \nabla_k L]$. This jump corresponds to a non-zero value of a surface stress-energy tensor on the boundary [11]. From Eq. (19), we obtain $[m^k \nabla_k L] = -8\pi \frac{LP}{\kappa}$. Taking the limit of $B' \rightarrow B$ from the outside, Eq. (22) becomes

$$M = -2 \oint_B d^2y \sqrt{\sigma} \frac{LP}{\kappa} + T_* \int_V d^3x \sqrt{h} \frac{\rho + 3P}{T}, \quad (23)$$

In deriving Eq. (23), we employed the equation $4\pi(\rho + 3P)L = \nabla_k \nabla^k L$, that follows from $S_{ij} h^{ij} = 0$, and Tolman's law, Eq. (11). Using Eq. (12), we find

$$2 \int_V d^3x \sqrt{h} \frac{\rho}{T} = 3\Omega - \frac{M + 2 \oint_B d^2y \sqrt{\sigma} LP/\kappa}{T_*}. \quad (24)$$

For a fluid with a linear equation of state $P = \gamma\rho$, where γ is a constant, $\int_V d^3x \sqrt{h} \rho/T = \Omega/(1 + \gamma)$. Then, Eq. (24) becomes

$$\Omega = \frac{1 + \gamma}{1 + 3\gamma} \frac{M + 2 \oint_B d^2y \sqrt{\sigma} LP/\kappa}{T_*}. \quad (25)$$

Eq. (25) demonstrates explicitly how Ω is expressed solely in terms of variables that are defined on the boundary B . For spherical symmetric systems, Eq. (25) reproduces the results of Ref. [2].

For a general equation of state, a closed expression for Ω in terms of the Komar mass is not possible, but we can identify an upper and a lower bound. For a fluid that satisfies the weak and dominant energy conditions ($0 \leq P \leq \rho$), $\Omega/2 \leq \int_V d^3x \sqrt{h} \rho/T \leq \Omega$. Hence, by Eq. (24)

$$\frac{1}{2} \leq \frac{\Omega T_*}{M + 2 \oint_B d^2y \sqrt{\sigma} LP/\kappa} \leq 1. \quad (26)$$

For a boundary that corresponds to a stellar surface, $P = 0$, and Eq. (26) simplifies, $\frac{M}{2T_*} \leq \Omega \leq \frac{M}{T_*}$.

The results above suggest that the Komar mass is a strong candidate for the internal energy in a self-gravitating system. However, at the moment this assertion is only tentative: we have found no natural splitting of the one-form Θ into internal energy and work. In any case, the internal energy is not an independent variable in the thermodynamic state space of self-gravitating systems—unlike ordinary thermodynamics—but a function of the more fundamental geometric variables σ_{ij} and κ_{ij} .

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